Lec 2: 08/26/2010

Necessary Conditions to Initiate Nuclear Burning:

As we saw, the source of Sun energy cannot be gravitational contraction alone. The key player is nuclear burning. Lets make estimates of the necessary mass and radius for nuclear burning to start.

We consider a spherical cloud of ionized Hydrogen at temperature T. For simplicity, we assume uniform density. If N is the total number of protons (the same for electrons), the total mass M will be Nm_p (m_p being proton mass). The number density of protons n (the same for electrons) is $\frac{N}{\frac{4\pi}{3}R^3}$ (R being the radius of cloud).

As a rough approximation, the average kinetic energy of a particle should be equal to its gravitational potential energy to avoid gravitational collapse.

The gravitational potential energy per particle at radius R is given by:

$$E_g = \frac{G m_p^2 N}{R} = \left(\frac{4\pi}{3}\right)^{1/3} G m_p^2 N^{2/3} n^{1/3}$$

The average kinetic energy is:

$$K = \frac{3}{2} k_B T + E_F \qquad (k_B: \text{Boltzmann constant})$$

The first term on the right-hand side is thermal energy at temperature T. The second term is Fermi energy, which is purely quantum mechanical. Note that classically particles at zero temperature have no kinetic energy. However, the zero point energy of a particle is nonzero in quantum mechanics (recall particle in a box).

Protons and electrons are both fermions. We know from Pauli blocking that two or more identical fermions cannot occupy the same eigenstate. This implies that a number N of electrons will occupy energy levels up to a maximum value E_F (called Fermi energy) at zero temperature. It can be shown that:

$$E_F = \frac{\hbar^2}{2m_e}(3\pi^2 n)^{2/3} \quad \left(\hbar = \frac{h}{2\pi},\ h\text{: Planck constant; } m_e\text{: electron mass}\right)$$

Therefore, in order to avoid gravitational collapse, we need to have:

$$\frac{3}{2}k_B T + \frac{\hbar^2}{2m_e}(3\pi^2 n)^{2/3} \sim G\, m_p^2\, N^{2/3}\, n^{1/3}$$

Note that we have neglected Fermi energy of protons because it is much smaller than that of electrons ($m_p \simeq 1000\, m_e$). In the case that there are no electrons and the number density of protons is very high, Fermi energy of protons will be important (like what happens in neutron stars).

We can find temperature T as a function of number density n (note that N is a constant):

$$k_B T \sim G m_p^2 N^{\frac{2}{3}} n^{\frac{1}{3}} - \left(\frac{3\pi^2}{2}\right)^{\frac{2}{3}} \frac{\hbar^2}{m_e} n^{\frac{2}{3}}$$

Here we have ignored numerical factors of order 1.

At larg radius number density is small, and hence the first term on the right-hand side dominates. As cloud contracts n, thus T, increase. The second term becomes important at some point and T reaches a maximum value T_{max}. Beyond that point (i.e. for smaller radius) T will decrease.

The number density at which T reaches its maximum value is:

$$n_c^{\frac{1}{3}} \sim \frac{\alpha_G}{(3\pi^2)^{2/3}} \frac{N^{\frac{2}{3}}}{\lambda\!\!\!^{-}_e} \quad \left(\alpha_G \equiv \frac{G m_p^2}{\hbar c} \simeq 6 \times 10^{-39},\ \lambda\!\!\!^{-}_e \equiv \frac{\hbar}{m_e c} \simeq 3.8 \times 10^{-11}\,(cm)\right)$$

gravitational equivalent of fine structure Constant

Compton wavelength of electron

Maximum temperature is:

$$k_B T_{max} \sim \frac{\alpha_G^2}{2(3\pi^2)^{2/3}} N^{\frac{4}{3}} m_e c^2 \quad *$$

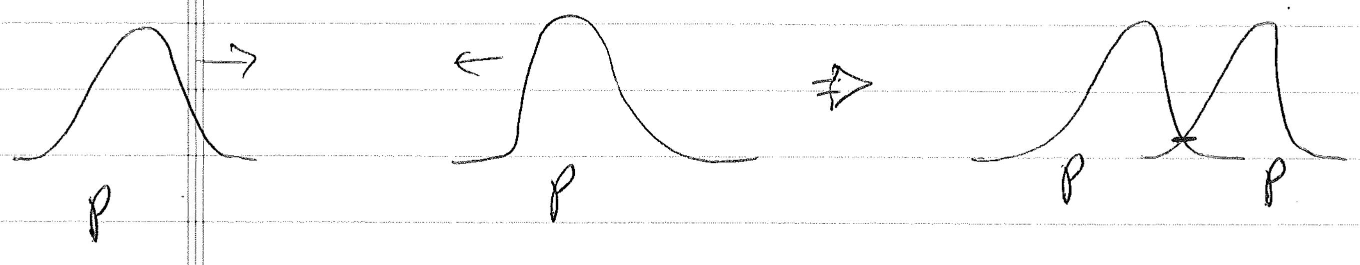
We note that temperature T comes as a result of nuclear burning. Therefore, in order to initiate and sustain burning, T_{max} must be high enough to bring two protons sufficiently close to each other so that $p+p \rightarrow {}^2D$ can occur. In other words, kinetic energy must be high enough to overcome Coulomb repulsion between protons and bring them to a sufficiently small distance where the short range nuclear force can successfully lead to pp fusion.

What is needed is for protons to be close enough to each other to have an overlap of their wavepackets. The typical width of wavepackets is represented by the deBroglie wavelength of particle;

$$\lambda_{deB} \equiv \frac{\hbar}{m_p v} \quad (v;\ \text{velocity})$$

∴ We do not need 100% overlap of the wavepackets.

As long as the wavepackets are within a distance $\sim \lambda_{deB}$ fusion can happen:



We will discuss this in detail later on. As we will see this is a tunneling problem for a potential consisting of short range attractive nuclear force and long range repulsive Coulomb force.

For the time being, lets assume a 10% overlap of the wavepackets. We then need:

$$k_B T_{max} \gtrsim 0.1\, \alpha^2 m_p c^2 \qquad \left(\alpha \equiv \frac{e^2}{\hbar c}\right)$$

fine structure constant

From equation $*$ we find the condition:

$$\boxed{N > N_* \equiv 4\times 10^{56} \Rightarrow M > M_* = N_* m_p \sim 4\times 10^{32}\, g}$$

We note that $\underline{\underline{M_* \sim 0.1 M_\odot}}$. It is remarkable that we find the lower bound on the cloud mass required to initiate Hydrogen burning that is in agreement with the minimum mass of observed stars.

The minimum radius is approximately found to be:

$$R_* \sim \frac{G M_* m_p}{k_B T_{max}} \sim \boxed{3 \times 10^{10}\ \mathrm{cm}}$$

Note that $\underline{\underline{R_* \sim \frac{1}{2} R_\odot}}$.

Our discussion has focused on Hydrogen burning (called the main sequence). Once all Hydrogen (at the core) is consumed to produce Helium the main sequence ends. The next reaction will be Helium burning to produce ^{12}C and ^{16}O. But Helium burning requires higher temperature and density (because of a larger Coulomb repulsion).

It can only start after the core contracts further. What happens after the end of Helium burning depends on the mass.

For masses below the Chandrsekhar limit ($M_{Ch} \simeq 1.5 M_\odot$) there will be no nuclear burning beyond that of Heliom. The reason being that pressure from degenerate electron gas is sufficient to prevent gravitational collapse of ^{12}C, ^{16}O core. For masses larger than the Chandrsekhar limit nuclear burning will continue. It will finally stop at ^{56}Fe, beyond which it cannot occur. At this point the star has an onion-shaped structure with heavier elements being at the inner layers. The gravitational contraction of the ^{56}Fe core can result in formation of a proton star (which is stable because of the pressure from a degenerate neutron-proton gas) and supernova explosion, or formation of a black hole. We will discuss all these scenarios in detail later on.